Stat 165 problem for CWE August 2004

(1) Suppose $X_1, \ldots, X_n$ are iid samples from a common discrete distribution

$$f(x|\theta) = P_\theta(X_1 = x) = \frac{e^{\theta x}}{e^{-\theta} + 1 + e^{\theta}}, \quad x = -1, 0, 1,$$

where θ is a real-valued unknown parameter. We write $X^n = (X_1, \ldots, X_n)$.

(a) Find a 1D minimal sufficient statistic for θ based on X^n.

(b) Find a MLE $\hat{\theta}$ for θ based on X^n.

(c) Show that $\hat{\theta}$ is asymptotically efficient and give the expressions of the asymptotic mean and variance of $\hat{\theta}$.

(d) For large n and $\alpha \in (0,1)$, construct a level $1-\alpha$ confidence interval for θ based on X^n.

(e) Let θ follow a (prior) double exponential distribution with density $\pi(\theta) = \frac{1}{2}\, e^{-|\theta|}$, $\theta \in \mathbb{R}$. Conduct a Bayesian test based on a single observation $X_1 = 1$ for the hypotheses $H_0 : \theta < 0$ vs $H_1 : \theta > 0$.

(2) For $n = 1, 2, \ldots$, let $\{\epsilon_n\}$, $\{X_n\}$ and $\{Y_n\}$ be three independent sequences of iid random variables with $X_1 \sim N(\mu_1, 1)$, $Y_1 \sim N(\mu_2, 1)$, and

$$P(\epsilon_1 = 0) = P(\epsilon_1 = 1) = 1/2,$$

where μ_1 and μ_2 are unknown parameters. For $j = 1, \ldots, n$, suppose the vector (ϵ_j, X_j, Y_j) cannot be observed; instead, the mixture

$$W_j = \epsilon_j X_j + (1 - \epsilon_j) Y_j$$

is observed. We write $W^n = (W_1, \ldots, W_n)$.

(a) Construct consistent estimators T_{1n} and T_{2n} for μ_1 and μ_2 respectively based on W^n. Hint: Use the method of moments.

(b) Conduct a test for $H_0 :\ \mu_1 = \mu_2$ vs $H_1 :\ \mu_1 \neq \mu_2$ based on W^n. Hint: Express the difference $\mu_2 - \mu_1$ as a function of the variance of W_1, say σ^2.