Statistics 164 Comprehensive Written Exam
August, 2004

1. Let U_1, U_2 be independent $\text{Exp}(\lambda)$ random variables.

   a. Find the distribution of $W = \min(U_1, U_2)$. Show your work.

   b. Find the density of $X = U_1 + U_2$. Show your work.

   c. Let $Y \sim U(0,1)$ be independent of U_1, U_2. Find the distribution of $Z = XY$.

2. Recall that the entropy of a discrete random variable X with probability mass function $p_i = P(X = x_i)$, $i = 1, \ldots, N$, is given by $H(X) = -\sum_{i=1}^{N} p_i \log_2 p_i$. Show directly that $0 \leq H(X) \leq \log_2 N$.

3. Suppose that $X_1, X_2, \ldots, X$ all have finite second moments. Show that $E(X_n - X)^2 \to 0$ implies that $X_n \to X$ in probability.

4. Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 1$ and finite variance σ^2. Let $Y_n = n^{-1} \sum_{i=1}^{n} X_i$.

   a. What does the ordinary CLT say about $\{Y_n\}$? Answer precisely.

   b. What can you say about the limiting distribution of $W_n = e^{\lambda Y_n}$ after suitable normalization? Show your work.

   c. What can you say about the limiting distribution of $Z_n = (Y_n - 1)^2 + (Y_n - 1)^3$ after suitable normalization? Show your work.

5. Let $\mathbf{X} \sim N_p(\mu, I)$, and let A, B be symmetric $p \times p$ matrices such that $AB = 0$. We wish to show that the quadratic forms $Y_1 = \mathbf{X}^t A \mathbf{X}$ and $Y_2 = \mathbf{X}^t B \mathbf{X}$ are independent.

   a. Show that $BA = 0$. Therefore A and B commute.

It follows from (a) that there exists an orthogonal matrix Γ such that $\Lambda_1 = \Gamma^t A \Gamma$ and $\Lambda_2 = \Gamma^t B \Gamma$ are diagonal.

   b. Show that $\Lambda_1 \Lambda_2 = 0$.

   c. Let $\mathbf{Z} = \Gamma^t \mathbf{X}$. What is the distribution of $\mathbf{Z}$?

   d. Use (b) and (c) to show that Y_1 and Y_2 are independent.

6. Let $X_2, X_2, \ldots, X$ and $Y_2, Y_2, \ldots, Y$ be random variables defined on the same probability space. Note that $X_n \to X$ in probability if and only if $X_n - X = o_p(1)$. Using basic fact about the stochastic order relations O_p and o_p, or by direct arguments, establish the following. In either case, show your work.

a. If $X_n \to X$ in probability and $Y_n \to Y$ in probability, then $X_n + Y_n \to X + Y$ in probability.

b. If $X_n \to X$ in probability and $Y_n \to Y$ in probability, then $X_n Y_n \to XY$ in probability.

Now let $W_1, W_2, \ldots$ be i.i.d. with finite variance σ^2. Define

$$\overline{W}_n = \frac{1}{n}\sum_{i=1}^{n} W_i \qquad S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(W_i - \overline{W}_n)^2$$

c. Use (a) and (b) to show that $S_n^2 \to \sigma^2$ in probability.