COMPREHENSIVE WRITTEN EXAMINATION, PAPER I

FRIDAY AUGUST 16, 2004, 9:00 A.M.–1:00 P.M.

STATISTICS 155 QUESTION

Problem 1:

(a) Define four common types of convergence for sequences of random variables and explain clearly and precisely the relationships between them.

(b) State Kolmogorov's "Three Series Theorem" and two forms of the strong law of large numbers.

(c) Let ξ_n be independent random variables ($n = 1, 2 \ldots$) taking values $\pm\frac{1}{n}$ with $P(\xi_n = \frac{1}{n}) = \mathrm{P}_n$. Obtain the probability of convergence of $\sum \xi_n$ as a function of P_n. (Hint: Consider $\sum(\xi_n - \epsilon\xi_n)$.) Specialize to the case $\mathrm{P}_n = \frac{1}{2}$ for all $n \geq$ some N_0.

Problem 2:

(a) Define the characteristic function (c.f.) of a random variable ξ and show that it always exists.
(b) State the Common Inversion Formula applicable when the c.f. is in $L_1(-\infty, \infty)$.
(c) State the Continuity Theorem for c.f.'s in as much generality as you can.
(d) Evaluate the c.f. of a Poisson random variable with mean m.
(e) Let $\phi(t)$ be the c.f. of an integer valued random variable ξ. Show that

$$P(\xi = n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(t)e^{-int}dt \qquad n = 0, \pm 1 \ldots.$$

(Be precise!)

Sketch of solution for Problem 1: Show $\sum \mathrm{var}(\xi_n) < \infty$ so that $\sum(\xi_n - E\xi_n)$ converges a.s. Therefore $\sum \xi_n$ converges iff $\sum E\xi_n$ converges i.e. iff $\sum \frac{P_n - \frac{1}{2}}{n}$ converges. Therefore $P(\sum \xi_n \text{ converges}) = 1$ or 0 according as $\sum \frac{P_n - \frac{1}{2}}{n}$ converges or diverges.

Sketch of solution for problem 2: (a)-(c) should be known and (d) standard.

(e):

$$\begin{aligned}
\phi(t) &= \sum_{-\infty}^{\infty} P_n e^{int} \text{ so that } \int_{-\pi}^{\pi} \phi(t) e^{-int} = \int_{-\pi}^{\pi} \sum P_m e^{i(m-n)t} dt \\
&= \sum P_m \int_{-\pi}^{\pi} e^{i(m-n)t} dt \text{ by Fubini } \left(\sum P_m < \infty\right) \\
&= 2\pi P_n \text{ since } \int_{-\pi}^{\pi} e^{i(m-n)t} dt = 0 \text{ for } m \neq n
\end{aligned}$$