Stat 154 questions

Problem 1. Let μ_1, μ_2 be two finite measures on a measurable space $(X, \mathcal{S})$. Let also $\mathcal{S} \supset \mathcal{E}$ be a class of sets closed under finite intersections (the so-called π-class). Suppose that $\mu_1(A) = \mu_2(A)$ for all $A \in \mathcal{E}$ and that $X \in \mathcal{E}$. Show that $\mu_1(A) = \mu_2(A)$ for all $A \in \sigma(\mathcal{E})$.

Problem 2. Let f_n be a sequence of measurable function on a measurable space (X, S). (a) If $f_n \to f$ everywhere, show that f is a measurable function. (b) If the series $\sum_{n=1}^{\infty} f_n$ converges everywhere, show that $\sum_{n=1}^{\infty} f_n$ is measurable.

Problem 3. Without providing any proofs, describe the steps in the construction of the Lebesgue measure on the real line (including the completion step).

Problem 4. Let μ be the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Consider a sequence of Borel functions $f_n(x) = n^2 e^{-nx^2}$, $x \in \mathbb{R}$, and let $f(x) = 0$, $x \in \mathbb{R}$. Does f_n converge to f (a) for all x? (b) a.e. (μ)? (c) uniformly on $\mathbb{R}$? (d) in measure μ? (e) in $L^p(\mathbb{R}, \mu)$, $p \in [0, \infty)$?

Problem 5. Provide an example of two continuous Lebesgue-Stieltjes measures μ_1 and μ_2 on $(\mathbb{R}, \mathcal{B})$ such that μ_1 and μ_2 are not singular, μ_1 is not absolutely continuous with respect to μ_2, and μ_2 is not absolutely continuous with respect to μ_1. According to the Lebesgue decomposition theorem, $\mu_1 = \mu_{1,c} + \mu_{1,s}$ where $\mu_{1,c} \ll \mu_2$ and $\mu_{1,s} \perp \mu_2$. Compute the Radon-Nikodym derivative $d\mu_{1,c}/d\mu_2$.

Problem 6. Let $\mu = \nu$ be counting measures on $X = Y = \{1, 2, 3, \dots\}$. If

$$f(x,y) = \begin{cases} 2 - 2^{-x}, & \text{if } x = y, \\ -2 + 2^{-x}, & \text{if } x = y + 1, \\ 0, & \text{otherwise,} \end{cases}$$

show that the iterated integrals $\int_Y \{\int_X f(x,y)\mu(dx)\}\nu(dy)$ and $\int_X \{\int_Y f(x,y)\nu(dy)\}\mu(dx)$ exist but are not equal. Why does this not contradict Fubini's Theorem?