Stat 165 problem for CWE August 2003

Note: Completion of any six parts from (a) — (g) will earn the full score.

Let $\mathcal{E}(\lambda, \theta)$ denote the exponential distribution over (θ, ∞) with parameters $\lambda > 0$ and $\theta \in I\!R$. The density is expressed as

$$f(x|\lambda, \theta) = \lambda e^{-\lambda(x-\theta)}, \quad x \geq \theta.$$

Note that the standard exponential distribution is a special case with $\theta = 0$. Suppose $X_1, \ldots, X_n$ are iid samples from a population $\mathcal{E}(\lambda, \theta)$. We write $X^n = (X_1, \ldots, X_n)$.

(a) Find a minimal sufficient statistic for (λ, θ) based on X^n.

(b) Find a MLE $(\hat{\lambda}, \hat{\theta})$ for (λ, θ) based on X^n.

(c) Assume $\lambda = 1$ and θ is unknown. Consider a class of confidence intervals for θ that have the form $(X_{(1)} - c_1,\ X_{(1)} - c_2)$, where $0 \leq c_2 < c_1$ are two constants and $X_{(1)} = \min\{X_1, \ldots, X_n\}$. For fixed confidence level $1 - \alpha$, find c_1 and c_2 such that the confidence interval length $c_1 - c_2$ is minimized.

(d) Let $\lambda = 1$, and θ follow a (prior) uniform distribution over the interval $[0, 1]$. Under the squared error loss, show that

$$T_n = \frac{1}{n}\left(\frac{\xi_n e^{\xi_n}}{e^{\xi_n} - 1} - 1\right)$$

is the Bayes estimator for θ based on X^n, where $\xi_n = \min\{nX_{(1)}, n\}$.

(e) Let $\lambda = 1$, and θ follow a (prior) uniform distribution over the interval $[0, 1]$. Conduct the Bayesian test based on $n = 10$ observations with $X_{(1)} = 0.4$ for the hypotheses $H_0 : \theta \leq 0.35$ vs $H_1 : \theta > 0.35$.

(f) Let $\lambda = 1$ and consider $\theta \in [0, 1]$ to be an unknown parameter that need not follow a prior distribution. Show that T_n given in (d) is a consistent estimator of θ. **Hint:** Consider the cases $\theta = 0$ and $0 < \theta \leq 1$ separately for which ξ_n may have different asymptotic behaviors.

(g) Let $X_1, \ldots, X_n$ be iid samples from the population $\mathcal{E}(\lambda_1, 0)$, and $Y_1, \ldots, Y_n$ iid samples from the population $\mathcal{E}(\lambda_2, 0)$, i.e. both populations are standard exponential distributions. Also assume X^n and Y^n are independent. For a fixed significance level $\alpha \in (0, 1)$ and with large sample size n, develop a test for $H_0 : \lambda_1 = \lambda_2$ vs $H_0 : \lambda_1 \neq \lambda_2$.