STAT 164 Question – August 2003

Let $X, Y, Z$ be independent random variables with a common "St. Petersburg distribution":

$$P(X = 2^j) = P(Y = 2^j) = P(Z = 2^j) = \frac{1}{2^j}, \quad \text{for } j = 1, 2, \ldots.$$

Further, let

$$(X', Y') = \begin{cases} (Y/2, X) & \text{if } X < Y, \\ (X, XZ) & \text{if } X = Y, \\ (X/2, Y) & \text{if } X > Y. \end{cases}$$

(a) Show that $X'$ and $Y'$ are independent St. Petersburg random variables.

(b) Show that

$$\frac{X + Y}{2} = X' + \frac{1}{2} Y' \, I(Y' \leq X'),$$

where $I(A)$ denotes the indicator function of the event $A$. Thus the sample average $(X + Y)/2$ for the St. Petersburg random variables $X$ and $Y$ is always as large as the St. Petersburg random variable $X'$, and sometimes larger! (This phenomenon could not happen if $X$ had a finite mean.)

(c) Compute the expected excess

$$E\left(\frac{X + Y}{2} - X'\right).$$