Stat 154 Comprehensive Exam 2003

1. Let $\{f_n\}$ be a sequence of measurable maps from a probability space $(\Omega, \mathcal{F}, P)$ to $(I\!R, \mathcal{B}(I\!R))$. Suppose that the sequence $\{f_n\}$ is *uniformly integrable.*

(i) **(5 Points.)** Say what it means for the sequence $\{f_n\}$ to be uniformly integrable.

(ii) **(10 Points.)** Suppose that $f_n \to f$ a.e. Show that f is integrable.

(iii) **(5 Points.)** Show that one can always find a sequence $\{\alpha_n\}$ in $I\!R_+$ such that $\alpha_n \to \infty$ as $n \to \infty$ and

$$P(|f| = \alpha_n) = 0$$

for all $n \in I\!N$.

(iv) **(10 Points.)** Show that if $\alpha \in I\!R_+$ is such that $P(|f| = \alpha) = 0$ then $f_n^\alpha \to f^\alpha$, a.e., where

$$f_n^\alpha \doteq f_n 1_{|f_n|<\alpha}, \quad f^\alpha \doteq f 1_{|f|<\alpha}.$$

(v) **(10 Points.)** Show that $f_n \to f$ in $L^1(P)$.

2.

(i) **(10 Points.)** State Fubini's theorem (for σ-finite measure spaces) precisely.

(ii) **(10 Points.)** Show that if F is a continuous distribution function then

$$\int_{-\infty}^{\infty} F(x) dF(x) = \frac{1}{2}.$$

Say precisely where the continuity of the function F is used.

(iii) **(5 Points.)** Say what is a counting measure on $I\!N$ (the set of of positive integers). Write the corresponding measure space explicitly.

(iv) **(10 Points.)** Let $\{f_n\}$ be a sequence of measurable maps from a σ finite measure space $(\Omega, \mathcal{F}, m)$ to $(\overline{I\!R}, \mathcal{B}(\overline{I\!R}))$. Suppose that

$$\sum_{n=1}^{\infty} \int_\Omega |f_n| \, dm < \infty.$$

Show that $\sum_{n=1}^{\infty} f_n$ converges absolutely, almost everywhere. Further show that the limit (i.e. the infinite sum) is integrable and

$$\int_\Omega \sum_{n=1}^{\infty} f_n \, dm = \sum_{n=1}^{\infty} \int_\Omega f_n \, dm.$$

3. Let μ, ν and ρ be finite measures on $(\Omega, \mathcal{F})$.

(i) **(5 Points.)** State the Radon-Nikodym theorem precisely.

(ii) **(10 Points.)** Show that if $\nu << \mu$ and $\mu << \rho$, then $\nu << \rho$. Furthermore

$$\frac{d\nu}{d\rho} = \frac{d\nu}{d\mu}\ \frac{d\mu}{d\rho}.$$

(iii) **(15 Points.)** Suppose that $\mu << \rho$ and $\nu << \rho$. Let

$$A \doteq \{\omega \in \Omega : \frac{d\nu}{d\rho} > 0 \quad \text{and} \quad \frac{d\mu}{d\rho} = 0\}.$$

Show that $\nu << \mu$ if and only if $\rho(A) = 0$ and in that case

$$\frac{d\nu}{d\mu} = 1_{\frac{d\mu}{d\rho} > 0}\ \frac{d\nu}{d\rho}\ \left(\frac{d\mu}{d\rho}\right)^{-1}.$$